

# Retarded/Advanced Correlation Functions and soft photon production in the Hard Loop Approximation

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## Abstract

We apply the retarded/advanced formalism of real time field theory to the QED or QED like case. We obtain a general expression for the imaginary part of the two-point correlation function in terms of discontinuities. The hard loop expansion is derived. The formalism is used to extract the divergent part of the soft fermion loop contribution to the real soft photon production.

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## I. Introduction

In the study of field theory at high temperature [1], the development of resummation techniques [2-4] has considerably extended the domain of validity of the perturbative approach. From the practical point of view, several possible signals for the formation of the quark-gluon plasma have now been studied in this framework. In particular the production of soft virtual photons [5,6] as well as hard real photons [7,8] have been discussed in great detail. Besides providing a consistent perturbative expansion, the resummed series gives a finite result for quantities which would otherwise not be well defined in a more naive approach.

The resummation technique of “hard loop” expansion has been formulated in the imaginary-time formalism (ITF) [1,9,10] of thermal field theories. On the other hand, there has been many developments in the real-time formalism (RTF) [1,10-12] or in the thermo-field dynamics (TFD) approach [13]. The latter seems more appropriate for the study of dynamical phenomena since time is kept as an independent variable throughout the calculation. Equivalently, in momentum space, no analytic continuation has to be performed to obtain physical quantities. Several studies have been devoted to the comparison of the ITF and the RTF [14-20]. In particular, a simple way to relate the two formalisms consists in constructing in the RTF the retarded/advanced (R/A) Green’s functions which, at least for the two- and three-point functions, have a close connection to the ITF Green’s functions. In this paper, we pursue the study of the R/A functions and show, in QED, how to formulate the hard-loop expansion: several formulae are derived which are useful in the calculation of physical quantities.

In the R/A formulation of the RTF, the propagators are still  $2 \times 2$  matrices but they are defined to be diagonal and constructed, at least at the lowest order of perturbation theory, from the retarded and advanced propagators of the  $T = 0$  theory [17,18]. As for the vertices, they become temperature dependent. The R/A (amputated) Green’s functions are in fact related to the causal combinations of RTF Green’s functions first introduced by Kobes [14,15]. In TFD, which has very similar Feynman rules to the RTF, such a diagonalization can also be performed and it is related to a thermal Bogoliubov transformation, thereby allowing the interpretation of the diagonal propagators as those of statistical quasi-particles [21]. This formulation seems therefore quite naturally appropriate to the study of thermal field theories.

In the following, we first complete the study started in [17] by constructing explicitly the diagonalization for fermion propagators and defining the Feynman rules in the R/A approach. We then discuss, in QED, the one loop expression for the 2-, 3- and 4-point functions in the context of the hard loop approximation and we easily recover the ITF results continued to real energies. We then set up a general formula, at the multi-loop level, for the 2-point functions and discuss how it is related to the ITF results. As an application, we consider the production of soft real photons and show that, despite cancellations due to thermal gauge invariance between various terms of the hard loop expansion, there survives a divergence. One of the advantages of the R/A approach over the ITF one is that it does not require any analytical continuation. Furthermore the spin structure of the diagrams is

the same as at  $T = 0$  and no continuation from Euclidian to Minkowski space is needed. Also, the possibility of using contour integration may simplify the calculation.

## II. The propagators and vertices in the R/A formalism.

In the real-time formalism, the fermion propagator, defined on a contour characterized by  $\sigma$  (fig. 1) can be written as a product of  $2 \times 2$  matrices [17].

$$S_F(P) = (\not{P} + M) U^F(P) \tilde{D}(P) V^F(P). \quad (1)$$

Denoting the retarded and advanced bosonic propagators at 0 temperature

$$\begin{aligned} \Delta_R(P) &= \frac{i}{P^2 - M^2 + i\varepsilon p_0} = \frac{i}{2\Omega} \left( \frac{1}{p_0 - \Omega + i\varepsilon} - \frac{1}{p_0 + \Omega + i\varepsilon} \right) \\ \Delta_A(P) &= \frac{i}{P^2 - M^2 - i\varepsilon p_0} = \frac{i}{2\Omega} \left( \frac{1}{p_0 - \Omega - i\varepsilon} - \frac{1}{p_0 + \Omega - i\varepsilon} \right), \end{aligned} \quad (2)$$

( $\Omega = (\vec{p}^2 + M^2)^{1/2}$ ) we introduce the diagonal matrix

$$\tilde{D}(P) = (\tilde{D}_{\alpha\beta}) = \begin{pmatrix} \Delta_R(P) & 0 \\ 0 & \Delta_A(P) \end{pmatrix} \quad (3)$$

The other matrices will be specified shortly after we define the photon propagator. In the Feynman gauge, the latter takes the form

$$G^{\mu\nu}(P) = -g^{\mu\nu} U^B(P) \tilde{D}(P) V^B(P) \quad (4)$$

where the same diagonal matrix as in eq. (1) appears. The diagonalization matrix, denoted generically  $U^{[\eta]}$ ,  $V^{[\eta]}$ , are defined in a similar fashion for bosons and fermions, and besides the contour  $\sigma$ , they depend on arbitrary scalar functions  $b(p)$ ,  $c(p)$

$$U^{[\eta]}(P) = (U_{i\alpha}^{[\eta]}) = -\eta n^{[\eta]}(-p_0) \begin{pmatrix} b^{-1} & \eta c^{-1} e^{(\sigma-\beta)p_0} \\ b^{-1} e^{-\sigma p_0} & c^{-1} \end{pmatrix} \quad (5)$$

$$V^{[\eta]}(P) = (V_{\alpha i}^{[\eta]}) = \begin{pmatrix} b & \eta b e^{(\sigma-\beta)p_0} \\ -c e^{-\sigma p_0} & -c \end{pmatrix} \quad (6)$$

where, not surprisingly,  $\eta = 1$  ( $[\eta] = B$ ) for a boson and  $\eta = -1$  ( $[\eta] = F$ ) for a fermion and  $n^{[\eta]}(p_0)$  is the usual Bose-Einstein or Fermi-Dirac distribution.  $\beta$  is the inverse of the temperature. The chemical potential has been set to 0, but it is easily introduced by the substitution  $\beta p_0 \rightarrow \beta(p_0 - \mu)$ ,  $\sigma p_0$  remaining invariant.

The basic principle, in the R/A formalism, is to associate the diagonalization matrices to the vertices in a natural way, keeping therefore the matrices  $\tilde{D}$  as diagonal propagators. This leads to different types of vertices depending on the momentum flow. For instance, for all incoming momenta as in fig. 2a ( $P + Q + R = 0$ ) we introduce

$$-i\gamma_{\alpha\beta\delta}(P, Q, R) = -ig_{abd} V_{\alpha a}^F(P) V_{\beta b}^B(Q) V_{\delta d}^F(R) \quad (7)$$

The latin indices refer to the 1 (particle) or 2 (ghost) fields of the usual formulation of RTF so that  $g_{111} = e$ ,  $g_{222} = -e$  ( $e$  is the electron charge), all the other couplings being 0. The greek indices take the values R or A. In defining the vertex function  $\gamma_{\alpha\beta\delta}$  we leave out the Dirac or Lorentz part of the vertex and keep only its scalar R/A structure. From the definitions above it can be shown that

$$\gamma_{\alpha\beta\delta}(P, Q, R) = e (b(P))^{\delta_{\alpha R}} (b(Q))^{\delta_{\beta R}} (b(R))^{\delta_{\delta R}} (-c(P))^{\delta_{\alpha A}} (-c(Q))^{\delta_{\beta A}} (-c(R))^{\delta_{\delta A}} e^{\sigma L_0} (1 - (-1)^{\delta_{\alpha R} + \delta_{\beta R}} e^{-\beta L_0}) \quad (8)$$

with  $L_0 = p_0 \delta_{\alpha R} + q_0 \delta_{\beta R} + r_0 \delta_{\delta R}$ .

It is immediately clear at this point that  $\gamma_{AAA}$  always vanishes while  $\gamma_{RRR} = 0$  because of momentum conservation. These results reflect the causality requirement that three particles propagating forward in time (or backward in time) cannot annihilate into (or be created from) the vacuum. It is convenient, and perhaps more natural to introduce the vertex in figure 2b, where the flow of momentum follows the fermion line. Momentum conservation now reads  $P + Q = R$ . We define

$$-i\gamma_{\alpha\beta;\delta}(P, Q, R) = -ig_{abd} V_{\alpha a}^F(P) V_{\beta b}^B(Q) U_{d\delta}^F(R) \quad (9)$$

An expression, similar to eq. (8), but not so symmetrical can be derived

$$\begin{aligned} \gamma_{\alpha\beta;\delta}(P, Q, R) = & -e (b(P))^{\delta_{\alpha R}} (b(Q))^{\delta_{\beta R}} (b(R))^{-\delta_{\delta R}} \\ & (-c(P)e^{-\sigma p_0})^{\delta_{\alpha A}} (-c(Q)e^{-\sigma q_0})^{\delta_{\beta A}} (-c(R)e^{-\sigma r_0})^{-\delta_{\delta A}} \\ & n^F(r_0)e^{\beta r_0 \delta_{\delta R}} \left[ (-1)^{\delta_{\delta R}} + (-1)^{\delta_{\alpha R}} e^{-\beta(p_0 \delta_{\alpha R} + q_0 \delta_{\beta R} - r_0 \delta_{\delta A})} \right] \end{aligned} \quad (10)$$

Admittedly, this expression is not very illuminating, but it immediately allows, by comparison with eq. (8) to derive the relations

$$\begin{aligned} \gamma_{\alpha\beta;R}(P, Q; R) &= -\frac{n^F(-r_0)e^{-\sigma r_0}}{b(R)c(-R)} \gamma_{\alpha\beta A}(P, Q, -R) \\ \gamma_{\alpha\beta;A}(P, Q; R) &= -\frac{n^F(r_0)e^{\sigma r_0}}{b(-R)c(R)} \gamma_{\alpha\beta R}(P, Q, -R) \end{aligned} \quad (11)$$

The obvious choice

$$b(-R) c(R) = -n^F(r_0) e^{\sigma r_0} \quad (12)$$

simplifies the “crossing” relation when the fermion momentum  $R$  is changed to  $-R$ , since we can then simply write

$$\gamma_{\alpha\beta;\delta}(P, Q; R) = \gamma_{\alpha\beta\bar{\delta}}(P, Q, -R) \quad (13)$$

where we have introduced  $\bar{\delta} = A, R$  the conjugate index of  $\delta = R, A$ . Similarly, if the “crossing” property of the photon line is studied, we find expression such as eq. (11) with

the difference that  $n^B(-q_0)$  appears rather than  $-n^F(-q_0)$ . The choice, in the bosonic diagonalization matrices

$$b(-R) c(R) = n^B(q_0) e^{\sigma q_0} \quad (14)$$

allows to keep the same crossing property eq. (13) for both fermion or boson lines. Further choices can be made to simplify the expression of the thermal vertices. For example, taking  $b \equiv 1$ , we immediately obtain

$$\begin{aligned} \gamma_{RRA}(P, Q, R) &= \gamma_{ARR}(P, Q, R) = \gamma_{RAR}(P, Q, R) = e \\ \gamma_{RAA}(P, Q, R) &= -e \frac{n^B(q_0) n^F(r_0)}{n^F(q_0 + r_0)} = -e (1 + n^B(q_0) - n^F(r_0)) \\ \gamma_{ARA}(P, Q, R) &= -e \frac{n^F(q_0) n^F(r_0)}{n^B(q_0 + r_0)} = -e (1 - n^F(q_0) - n^F(r_0)). \end{aligned} \quad (15)$$

Together with

$$\gamma_{RRR} = \gamma_{AAA} = 0 \quad (16)$$

and the temperature independent propagators

$$\begin{aligned} \tilde{S}_F(P) &= (\not{P} + M) \begin{pmatrix} \Delta_R(P) & 0 \\ 0 & \Delta_A(P) \end{pmatrix} \\ \tilde{G}^{\mu\nu}(P) &= -g^{\mu\nu} \begin{pmatrix} \Delta_R(P) & 0 \\ 0 & \Delta_A(P) \end{pmatrix} \end{aligned} \quad (17)$$

the diagrammatic rules for thermal QED in the R/A formalism are completely specified. Let us note that all reference to the arbitrary contour  $\sigma$  has now disappeared from the Feynman rules. Unlike eqs. (16), the results eqs. (15) and (17) are not universal. For instance, in [18] the propagators are chosen to be anti-diagonal so that reversal of momentum yields  $\tilde{D}(-P) = (\tilde{D}(P))^T$ . Here we keep the diagonal form as it appears more natural for the construction of loop diagrams. Furthermore, by a different choice of the arbitrary functions  $b$  and  $c$ , we could have imposed that the vertices of type  $\gamma_{RAA}$  be equal to the coupling  $e$ . The vertices such as  $\gamma_{RRA}$  would then have become linear in the statistical weights.

Any (amputated)  $n$ -point Green's function in the R/A formalism can be constructed from the corresponding ones in the RTF by a generalization of eqs. (7) and (10). The R/A functions then appear as specific linear combinations of the usual real-time functions. It is more practical, however, if one is interested in higher order perturbative calculations to construct the R/A functions directly by application of the rules given above. Despite a rather cumbersome notation, the Green's functions are nicely expressed in terms of tree diagrams.

The diagonalization approach to RTF is more than an algebraic trick. The R/A functions are “more natural” than the Green's functions in the real-time formalism. For example, it has been shown that the imaginary part of the 2-point function (there exists only two such functions related by complex conjugation) is proportional to the opacity factor in the Boltzman equation for a distribution near equilibrium [14,18]. Furthermore, the  $n$ -point

functions with all but one indices set equal to  $R$  are the causal Green's functions introduced by Kobes [14,15]. A different perspective comes from thermo-field dynamics. The Feynman rules in TFD and RTF are very similar (for systems in equilibrium which is the case considered here). Recently Henning and Umezawa [21] studied the diagonalization of the 2-point function in TFD. The diagonalization matrices (Bogoliubov transformations) have two parameters:  $\alpha$  which characterizes the thermal vacuum and  $s$ , with the relations to our free parameters given by  $\alpha = 1 - \sigma/\beta$  and  $s = \ln b\sqrt{1 + n(p_0)} = -\ln c\sqrt{1 + n(p_0)}$ . In TFD, the Bogoliubov matrices parametrize the transformations which take the physical (point-like) particles to the (thermal) quasi-particle states and the diagonal (un-amputated) 2-point function  $\tilde{D}(P)$  describes the propagation of the quasiparticles in the medium. We note also that by a different Bogoliubov transformation we could introduce the diagonal matrix constructed with the usual Feynman propagator and its complex conjugate.

We turn now to the study of loop diagrams.

### III. R/A functions in the one-loop approximation

Following the method introduced for  $\lambda\phi^3$  we construct 2-, 3- and 4-point functions perturbatively. We implicitly assume that we deal with QED or QCD. In general, we do not specify the spin structure of the diagrams since it factorizes from the R/A functions and it is the same as at 0 temperature. We first consider a generic two-point function with external momentum  $Q$  and index  $\beta$  flowing through the diagram. The internal momenta are  $P$ , with fermion or boson statistics  $\eta_P$ , and  $R$ , with statistics  $\eta_R$ , such that  $P + Q = R$  [fig. 3]. Applying the rules eqs. (15)-(17) and using the crossing relation eq. (13) we arrive at

$$\begin{aligned} -i\Gamma_{\beta\beta}(Q) &= -e^2 \int \frac{d^n P}{(2\pi)^n} \mathcal{P} \gamma_{\alpha\beta\bar{\delta}}(P, Q, -R) \Delta_{\delta}(R) \gamma_{\bar{\alpha}\bar{\beta}\delta}(-P, -Q, R) \Delta_{\alpha}(P) \\ &= -e^2 \int \frac{d^n P}{(2\pi)^n} \mathcal{P} \left[ \left( \frac{1}{2} + \eta_P n^{[\eta_P]}(p_0) \right) (\Delta_R(P) - \Delta_A(P)) \Delta_{\beta}(R) \right. \\ &\quad \left. + \left( \frac{1}{2} + \eta_R n^{[\eta_R]}(r_0) \right) (\Delta_R(R) - \Delta_A(R)) \Delta_{\bar{\beta}}(P) \right]. \end{aligned} \quad (18)$$

The symbol  $\mathcal{P}$  denotes the Dirac and spin structure of the diagram. For example, for the fermion self-energy we have, in Feynman gauge  $\mathcal{P} = -\gamma_{\nu} (R + M) \gamma^{\nu}$  if  $R$  is the fermion internal line. In arriving at the final form of eq. (18) we have dropped in the integrand terms of type  $\Delta_R(P)\Delta_R(R)$  or  $\Delta_A(P)\Delta_A(R)$  independent of the statistical weight. These terms have poles in the  $p_0$  complex energy plane only on one side of the real axis. By closing the  $p_0$  integration contour in the other half-plane they are seen to give a vanishing contribution to the 2-point function. By the same token, the factor  $\frac{1}{2}$  can be replaced by any constant because shifting the numerical value only generates irrelevant  $\Delta_R\Delta_R$  or  $\Delta_A\Delta_A$  factors. We prefer keeping the  $\frac{1}{2}$  factor since, using the relation

$$\Delta_R(P) - \Delta_A(P) = 2\pi \varepsilon(p_0) \delta(P^2 - M^2), \quad (19)$$

we find that it leads to an invariant form under the reversal of the sign of momentum  $P$ :

$$\left(\frac{1}{2} + \eta_P n^{[\eta_P]}(P)\right) (\Delta_R(P) - \Delta_A(P)) = 2\pi \left(\frac{1}{2} + \eta_P n^{[\eta_P]}(|p_0|)\right) \delta(P^2 - M^2). \quad (20)$$

When evaluating eq. (18) we can either use the  $\delta$ -function constraint or close the  $P_0$  contour in a conveniently chosen half-plane. For instance for the first term in  $\Gamma_{RR}(Q)$  encircling poles in the upper half-plane it is seen that only the poles of  $\Delta_A(P)$  contribute. In particular, the poles on the imaginary axis associated to the statistical function never contribute since they have a vanishing residue as  $\Delta_R(P) - \Delta_A(P) = 0$  on the imaginary axis. For the second term, it is preferable to close the contour in the lower-half plane to retain only the singularities of  $\Delta_R(R) = \Delta_R(P + Q)$ .

The two-point function obeys some general relations [17,18]. We have

$$\Gamma_{\alpha\alpha}^*(Q) = \Gamma_{\bar{\alpha}\bar{\alpha}}(Q) \quad (21)$$

provided, it is assumed that the complex conjugation does not operate on the spinor structure  $\mathcal{P}$  but only on the R/A part of the integrand. It is based on the property

$$\Delta_\alpha^*(P) = -\Delta_{\bar{\alpha}}(P). \quad (22)$$

We can also prove, for QED/QCD like theories and assuming massless particles, that

$$\Gamma_{\alpha\alpha}(-Q) = \pm \Gamma_{\bar{\alpha}\bar{\alpha}}(Q) \quad (23)$$

where the (+) sign refers to a bosonic external line and the (−) sign to a fermionic one. This equation can be obtained by reversing the sign of all momenta (external as well as internal) and using

$$\Delta_\alpha(-P) = \Delta_{\bar{\alpha}}(P) \quad (24)$$

$$\frac{1}{2} + \eta_P n^{[\eta_P]}(-p_0) = -\left(\frac{1}{2} + \eta_P n^{[\eta_P]}(p_0)\right). \quad (25)$$

Eqs. (21) and (23) can be shown to hold at any loop order. Let us remark as a consequence that, for massless fields, eq. (23) holds true for the full propagators *i.e.*, for propagators after the geometrical series of self energy corrections has been summed.

The other two point functions satisfy the obvious relation

$$\Gamma_{\alpha\bar{\alpha}}(Q) = 0 \quad (26)$$

which expresses the fact that a particle propagating forward (backward) in time cannot turn into a particle propagating backward (forward) in time by self-interactions. Technically, eq. (26) is satisfied because its integrand is a sum of terms having poles only one side of the real axis with no statistical weights attached to them.

Turning now to the 3-point function with the momenta as indicated in fig. 4 ( $P + Q + R = 0$ ) we can derive by the same method as above

$$\Gamma_{\alpha\alpha\alpha}(P, Q, R) = 0 \quad (27)$$

which is true to all order of perturbation theory. For the other functions we find it convenient to factorize out the tree vertex expression of the 3-point function and define (recall that  $\gamma_{\alpha\beta\delta}$  contains only the scalar part of the vertex as defined in eq. (15))

$$V_{\alpha\beta\delta}(P, Q, R) = \frac{\Gamma_{\alpha\beta\delta}(P, Q, R)}{\gamma_{\alpha\beta\delta}(P, Q, R)}, \quad (28)$$

not all indices being identical. It can be proven

$$\begin{aligned} V_{\alpha\beta\delta}(P, Q, R) = -e^2 \int \frac{d^n L_1}{(2\pi)^n} \mathcal{P} & \\ & \left[ \left( \frac{1}{2} + \eta_1 n^{[\eta_1]}(l_{10}) \right) (\Delta_R(L_1) - \Delta_A(L_1)) \Delta_\alpha(L_2) \Delta_{\bar{\delta}}(L_3) \right. \\ & + \left( \frac{1}{2} + \eta_2 n^{[\eta_2]}(l_{20}) \right) (\Delta_R(L_2) - \Delta_A(L_2)) \Delta_\beta(L_3) \Delta_{\bar{\alpha}}(L_1) \\ & \left. + \left( \frac{1}{2} + \eta_3 n^{[\eta_3]}(l_{30}) \right) (\Delta_R(L_3) - \Delta_A(L_3)) \Delta_\delta(L_1) \Delta_{\bar{\beta}}(L_2) \right]. \end{aligned} \quad (29)$$

The symbol  $\mathcal{P}$  is now the appropriate one for the 3-point function. Again, the constant factor 1/2 can be arbitrarily changed. We observe that eq. (29) appears as a sum of tree amplitudes since each  $\Delta_R - \Delta_A$  combination puts the corresponding line on shell with each cut line carrying a weight  $(\frac{1}{2} + \eta_i n^{[\eta_i]}(l_{i0})) \varepsilon(l_{i0}) \delta(L_i^2 - M^2)$  or  $\eta_i n^{[\eta_i]}(l_{i0}) \varepsilon(l_{i0}) \delta(L_i^2 - M^2)$  if the factor 1/2 is dropped in eq. (29). A graphical representation of this is given in fig. 5. As for the 2-point function we derive [17,18]

$$V_{\alpha\beta\delta}^*(P, Q, R) = V_{\bar{\alpha}\bar{\beta}\bar{\delta}}(P, Q, R) \quad (30)$$

$$V_{\alpha\beta\delta}(-P, -Q, -R) = \pm V_{\bar{\alpha}\bar{\beta}\bar{\delta}}(P, Q, R). \quad (31)$$

Where the last equation holds true for massless fields in the QED/QCD case. The + sign is appropriate for a fermion-antifermion-gauge boson coupling, while the - sign is for the triple gauge boson coupling.

Turning to the 4-point functions they can also be expressed in terms of tree diagrams. They are easily obtained from [17] with the appropriate modifications of the statistical factors, and taking into account the appropriate Dirac structure.

#### IV. R/A functions in QED and the hard loop expansion.

In this section we write explicitly the relevant  $n$ -point functions in QED and discuss Ward identities as well as the hard loop approximation.



After performing the loop energy integration in an  $n$ -point function, by use of the  $\delta$ -function or by a contour deformation, the temperature dependent terms take the form

$$\begin{aligned} \int \frac{d^n L}{(2\pi)^n} n(l_0) (\Delta_R(L) - \Delta_A(L)) F(l_0, \vec{l}; P, Q, \dots) \\ = \frac{1}{2} \int \frac{\omega^{n-3}}{(2\pi)^{n-1}} n(\omega) d\omega \hat{d}\hat{l} (F(\omega, \vec{l}; P, Q, \dots) + F(-\omega, \vec{l}; P, Q, \dots)) \end{aligned} \quad (32)$$

where  $\omega = |\vec{l}|$  and  $\hat{d}\hat{l}$  is the symbol for angular integration in  $n$ -dimensions ( $\hat{l} = \vec{l}/\omega$ ). We have assumed massless particles for simplicity but the following argument also holds true in the massive case. Keeping only relevant terms we have to evaluate in practical cases dimensional integrals of type

$$I_\eta^{(p)} = \int d\omega \omega^{n-3} n^{[\eta]}(\omega) \left(\frac{\omega}{m}\right)^p \quad (33)$$

where  $m$  is a mass scale typical of the external momenta components  $(p_0, \vec{p})$ ,  $(q_0, \vec{q})$ ... (all external variables are supposed to have comparable sizes). This can be re-expressed with the variable  $z = \omega/T$  as

$$I_\eta^{(p)} = T^2 \left(\frac{T}{m}\right)^p T^{-2\epsilon} \int_0^\infty dz \frac{z^{1+p-2\epsilon}}{e^z - \eta} \quad (34)$$

where  $n = 4 - 2\epsilon$ . These integrals are easily expressed in terms of  $\Gamma$  and Riemann  $\zeta$  functions and we find:

$$\begin{aligned} I_1^{(p)} &= T^2 \left(\frac{T}{m}\right)^p T^{-\epsilon} \Gamma(2 + p - 2\epsilon) \zeta(2 + p - 2\epsilon) \\ I_{-1}^{(p)} &= (1 - 2^{-1-p+2\epsilon}) I_1^{(p)} \end{aligned} \quad (35)$$

Defining “soft” momenta as those with all components of  $\mathcal{O}(eT)$ ,  $e \ll 1$ , then a Green’s function with soft external momenta behaves as

$$I_\eta^{(P)} \sim T^2 \left(\frac{1}{e}\right)^p. \quad (35a)$$

Clearly the dominant term comes from the largest value of  $p$ , *i.e.*, from terms in the integrand with the highest power of  $L$ , the loop momentum. In other words, the dominant behavior arises from the parts in the integrand which are leading when  $l_i \sim T$ , hence the name of hard loop approximation. This is the basis of the resummation approach of Braaten and Pisarski [2,3] which consists in taking account of all hard loops at any order of perturbation theory. Several examples are discussed below.

Consider the massless fermion self-energy in the Feynman gauge (see fig. 6). The general expression is (we simplify the notation by using  $\Sigma_\alpha$  instead of  $\Sigma_{\alpha\alpha}$ )

$$\begin{aligned} -i\Sigma_\alpha(P) &= +e^2 \int \frac{d^n L}{(2\pi)^n} \gamma_\nu (\not{P} + \not{L}) \gamma^\nu \left[ \left(\frac{1}{2} + n^B(l_0)\right) (\Delta_R(L) - \Delta_A(L)) \Delta_\alpha(P + L) \right. \\ &\quad \left. + \left(\frac{1}{2} - n^F(p_0 + l_0)\right) (\Delta_R(P + L) - \Delta_A(P + L)) \Delta_{\bar{\alpha}}(L) \right]. \end{aligned} \quad (36)$$

For a soft fermion ( $p \sim eT$ ), we can, according to the above discussion, neglect  $\vec{P}$  compared to  $\vec{L}$  and with a change of variable in the second term ( $P + L \rightarrow -L$ ) we arrive at

$$-i\Sigma_\alpha(P) = -2(1-\epsilon)e^2 \int \frac{d^{n-1}L}{(2\pi)^{n-1}} \frac{n^B(\omega) + n^F(\omega)}{2\omega} [\vec{L} \Delta_\alpha(P+L) + \vec{L}' \Delta_\alpha(P+L')] \quad (37)$$

with  $L = (\omega, \vec{l})$ ,  $L' = (-\omega, \vec{l})$ . Consider now  $\Delta_\alpha(P+L)$  for example,

$$\Delta_\alpha(P+L) = \frac{i}{2|\vec{p} + \vec{l}|} \left[ \frac{1}{p^0 + \omega - |\vec{p} + \vec{l}| + i\varepsilon_\alpha} - \frac{1}{p^0 + \omega + |\vec{p} + \vec{l}| + i\varepsilon_\alpha} \right] \quad (38)$$

(we use the convention  $\varepsilon_R = \varepsilon$  and  $\varepsilon_A = -\varepsilon$ ,  $\varepsilon > 0$ ). Under the assumption that  $P$  is soft, we can safely neglect the second term, which behaves as  $1/\omega^2$ , and keep only the first one which for large  $\omega$  reduces to

$$\Delta_\alpha(P+L) = \frac{1}{2\omega} \frac{i}{p_0 - \vec{p} \cdot \hat{l} + i\varepsilon_\alpha}, \quad \hat{l} = \frac{\vec{l}}{\omega}. \quad (39)$$

In the hard loop approximation, only one of the two poles of the R/A propagator is relevant, namely the pole associated to Landau damping [3]. The pole associated to particle production which is the only one at 0 temperature, is suppressed by powers of the coupling constant (or equivalently by factors in  $1/T$ ). The fermion self-energy then takes the simple form

$$\Sigma_\alpha(P) = \frac{(1-\epsilon)e^2}{2\pi^2} \int d\omega \omega^{1-2\epsilon} (n^B(\omega) + n^F(\omega)) \int \frac{1}{2} \frac{d\hat{L}}{(2\pi)^{1-2\epsilon}} \frac{\hat{L}}{P\hat{L} + i\varepsilon_\alpha} \quad (40)$$

where we have introduced the light-like vector  $\hat{L} = (1, \hat{l})$ . The dimensional and the angular part of the loop integration factorize, as is well-known [3,4] and we can write our final result:

$$\Sigma_\alpha(P) = m_{th}^2(\epsilon) \int \frac{1}{2} \frac{d\hat{L}}{(2\pi)^{1-2\epsilon}} \frac{\hat{L}}{P\hat{L} + i\varepsilon_\alpha}. \quad (41)$$

In the 4-dimensional limit, the mass term  $m_{th}^2(\epsilon)$  reduces to the thermal mass squared [22]

$$m_{th}^2 = \frac{e^2 T^2}{8} \quad (42)$$

a well-known result. Although not necessary here, we have worked in  $n$ -dimensions for later purposes. We immediately have the useful property

$$\Sigma_\alpha(-P) = -\Sigma_{\bar{\alpha}}(P). \quad (43)$$

The effective propagator is defined as

$$\begin{aligned} {}^*S_\alpha(P) &= \frac{i}{\not{P} - \Sigma_\alpha(P)} \\ &= i \frac{(\not{P} - \Sigma_\alpha(P))}{D_\alpha^2(P)} \end{aligned} \quad (44)$$

where, in the second form,  $D_\alpha^2(P)$  is a scalar which need not be specified at this point. If  $P$  is soft, of  $\mathcal{O}(eT)$ , eq. (41) immediately shows that  $\Sigma_\alpha(P)$  is also of  $\mathcal{O}(eT)$ , hence the necessity of using the resummed propagator eq. (44) rather than the bare one for a consistent calculation. If on the other hand  $P$  is hard, of  $\mathcal{O}(T)$ , it appears that  $\Sigma_\alpha(P)$  is of  $\mathcal{O}(e^2T)$  from eq. (41) and therefore the self energy gives a (negligible) correction of  $\mathcal{O}(e^2)$  to the bare propagator. In general, hard propagators need not be resummed in a leading order calculation [3]. We recall the antisymmetry property of the effective fermion propagator which is a consequence of eq. (43)

$${}^*S_\alpha(-P) = -{}^*S_{\bar{\alpha}}(P). \quad (44a)$$

It will be used later to simplify calculations.

We could calculate, using the same technique, the vacuum polarization diagram and obtain the scalar  $\pi_\alpha^T(Q)$ ,  $\pi_\alpha^L(Q)$ , *i.e.*, the transverse and longitudinal polarization functions [23] which are the analytic continuations  $\pi^L(q_0 + i\varepsilon_\alpha, \vec{q})$ ,  $\pi^T(q_0 + i\varepsilon_\alpha, \vec{q})$  of the imaginary time approach. Since we do not need these results in the following we do not go into details and turn now to the three-point function.

For the sake of completeness, we consider temporarily the case of massive fermions and work in a general gauge where the photon propagators is denoted  $\mathcal{P}^{\nu\rho}(L_1)$   $\Delta_\alpha(L_1)$ . We also consider the complete expression not assuming the hard loop approximation. It is easy to prove (in fig. 4 we assume  $Q$  to be the photon momentum and  $P$  the incoming fermion momentum):

$$\begin{aligned} V_{\alpha\beta\delta}^\mu(P, Q, R) &= -e^2 \int \frac{d^n L_1}{(2\pi)^n} \gamma_\nu (\not{L}_2 + M) \gamma_\mu (\not{L}_1 + M) \gamma_\rho \mathcal{P}^{\nu\rho}(L_1) \\ &\quad \left[ \left( \frac{1}{2} + n^B(l_{10}) \right) \varepsilon(l_{10}) \delta(L_1^2) \Delta_\alpha(L_2) \Delta_{\bar{\delta}}(L_3) \right. \\ &\quad + \left( \frac{1}{2} - n^F(l_{20}) \right) \varepsilon(l_{20}) \delta(L_2^2 - M^2) \Delta_\beta(L_3) \Delta_{\bar{\alpha}}(L_1) \\ &\quad \left. + \left( \frac{1}{2} - n^F(l_{30}) \right) \varepsilon(l_{30}) \delta(L_3^2 - M^2) \Delta_\delta(L_1) \Delta_{\bar{\beta}}(L_2) \right]. \end{aligned} \quad (45)$$

In order to prove that the usual Ward identity holds true at finite temperature we construct

the scalar  $Q_\mu V_{\alpha\beta\delta}^\mu(P, Q, R)$ . Making use of the following identities:

$$\begin{aligned}
(\not{L}_2 + M) \not{Q} (\not{L}_3 + M) \delta(L_2^2 - M^2) \Delta_\beta(L_3) &= i(\not{L}_2 + M) \delta(L_2^2 - M^2) \\
(\not{L}_2 + M) \not{Q} (\not{L}_3 + M) \delta(L_3^2 - M^2) \Delta_{\bar{\beta}}(L_2) &= -i(\not{L}_3 + M) \delta(L_3^2 - M^2) \\
(\not{L}_2 + M) \not{Q} (\not{L}_3 + M) \delta(L_1^2) \Delta_\alpha(L_2) \Delta_{\bar{\delta}}(L_3) & \\
&= i((\not{L}_2 + M) \Delta_\alpha(L_2) - (\not{L}_3 + M) \Delta_{\bar{\delta}}(L_3)) \delta(L_1^2)
\end{aligned} \tag{46}$$

we easily derive the following identity

$$Q_\mu V_{\alpha\beta\delta}^\mu(P, Q, R) = \Sigma_\alpha(P) - \Sigma_{\bar{\delta}}(-R). \tag{47}$$

This holds true not only for the  $e^2 T^2$  terms but also for the  $e^2 T$  and of course the constant ( $T = 0$ ) pieces. In the derivation, the thermal factors do not play any particular role. Knowing that, at  $T = 0$ , eq. (47) is satisfied to any loop order it can presumably be proven, by similar methods, that it is also true at finite temperature [24].

Let us now turn to the hard loop approximation. As is the case of the self-energy it is found that only the Landau damping contribution has to be considered. We are then justified in writing the propagators

$$\begin{aligned}
\Delta_\alpha(L + P) &= \frac{1}{2\omega} \frac{i}{P\hat{L} + i\varepsilon_\alpha} \\
\Delta_{\bar{\delta}}(L - R) &= \frac{1}{2\omega} \frac{-i}{R\hat{L} + i\varepsilon_{\bar{\delta}}},
\end{aligned} \tag{48}$$

and recombining terms proportional to  $(\frac{1}{2} - n_F)$  in eq. (46) we have (massless theory, Feynman gauge)

$$\begin{aligned}
V_{\alpha\beta\delta}^\mu(P, Q, R) &= -\frac{1}{2}(1 - \epsilon)e^2 \int \frac{d^n L}{(2\pi)^{n-1}} \hat{L} \gamma^\mu \hat{L} (n^B(l_0) + n^F(l_0)) \varepsilon(l_0) \delta(L^2) \\
&\quad \frac{1}{P\hat{L} + i\varepsilon_\alpha} \frac{1}{R\hat{L} + i\varepsilon_{\bar{\delta}}} \\
&= -m_{th}^2(\epsilon) \int \frac{1}{2} \frac{d\hat{L}}{(2\pi)^{1-2\epsilon}} \frac{\hat{L}^\mu \hat{L}}{(P\hat{L} + i\varepsilon_\alpha)(R\hat{L} + i\varepsilon_{\bar{\delta}})}
\end{aligned} \tag{49}$$

The interesting feature about this formula is its independence on the retarded or advanced prescription on the photon momentum  $Q$ : the analytic structure of the vertex is entirely given by the prescription on the fermion momenta. As a consequence, in the hard loop approximation we can write

$$V_{\alpha R\delta}^\mu(P, Q, R) - V_{\alpha A\delta}^\mu(P, Q, R) = 0 \tag{50}$$

(not all indices being equal in the above expression). Mathematically this relation expresses the fact that the function  $V_{\alpha\beta\delta}^\mu(P, Q, R)$  of three variables  $P, Q, R$ , considered to be independent has no singularity when crossing the real  $q_0$  axis. In other words

$$\text{Disc}_Q V_{\alpha R\delta}^\mu(P, Q, R) = 0 \tag{50a}$$

The independence of the variables  $P, Q, R$  is understood in the following sense: the retarded or advanced prescriptions on the momenta  $P$  and  $R$  respectively are not dependent on that of  $Q$  since  $\beta$  takes the values  $R, A$  without changing the indices  $\alpha$  and  $\delta$ . In the diagrammatic decomposition of fig. 5, eq. (49) is entirely represented by term a) with only the internal photon line being cut and the  $i\varepsilon$  prescription entirely carried by the internal fermion lines. This property results from a rearrangement of terms in the integrand and, as a consequence, the statistical weight attached to the cut photon line is not only  $2\pi(\frac{1}{2} + n^B(l_0)) \epsilon(l_0) \delta(L^2)$  as in the general case but rather  $2\pi(n^B(l_0) + n^F(l_0)) \epsilon(l_0) \delta(L^2)$  for the hard loop case. Since the angular dependence factorizes out in the integrand we can perform the integration over the energy and the length of the internal momentum variable and thus recover the thermal mass factor. This property will simplify the cut-structure of higher loop diagrams as will be seen later.

The contraction of the vertex function with  $Q_\mu$  immediately yields eq. (47), which now becomes a relation between Green's function evaluated in the hard loop approximation. Pseudo Ward identities are also obtained [4]

$$\begin{aligned} P \cdot V_{\alpha\beta\delta}(P, Q, R) &= -\Sigma_\delta(R) \\ R \cdot V_{\alpha\beta\delta}(P, Q, R) &= -\Sigma_\alpha(P) \end{aligned} \quad (51)$$

which are only true at the leading  $eT$  level. We define now an effective vertex

$$^*V_{\alpha\beta\delta}^\mu(P, Q, R) = \gamma^\mu + V_{\alpha\beta\delta}^\mu(P, Q, R). \quad (52)$$

A dimensional analysis of eq. (49) shows that if all external vertex momenta are soft, then the function  $V_{\alpha\beta\delta}^\mu$  is of  $\mathcal{O}(1)$  like the bare vertex and, in that case, the effective vertex  $^*V_{\alpha\beta\delta}^\mu$  has to be used for a consistent calculation. If, on the contrary, one (and therefore at least two) external momenta are hard then the loop correction is down by at least a factor  $e^2T$  compared to the tree vertex.

We turn now to the 4-point function and, as an example, we consider the case with 2 external photons and an fermion-antifermion pair. We restrict ourselves, for the moment, to functions of type  $C_{RRRA}^{\mu\nu}(P, Q_1, Q_2, R)$  and cyclic permutations on the R/A indices. They are the Fourier transforms of the retarded products of fields [14]. Defining the sum of diagrams of fig. 7 as  $ie^2 C_{\alpha\beta\gamma\delta}^{\mu\nu}(P, Q_1, Q_2, R)$  we derive the hard loop expression (the superscripts  $\mu$  and  $\nu$  refer to the Dirac indices),

$$\begin{aligned} C_{\alpha\beta\gamma\delta}^{\mu\nu}(P, Q_1, Q_2, R) &= m_{th}^2(\epsilon) \int \frac{1}{2} \frac{d\hat{L}}{(2\pi)^{1-2\epsilon}} \hat{L}^\mu \hat{L}^\nu \hat{L} \frac{1}{P\hat{L} + i\varepsilon_\alpha} \frac{1}{R\hat{L} + i\varepsilon_\delta} \\ &\quad \left( \frac{1}{(P + Q_1)\hat{L} + i(\varepsilon_\alpha + \varepsilon_\beta)} + \frac{1}{(P + Q_2)\hat{L} + i(\varepsilon_\alpha + \varepsilon_\delta)} \right) \end{aligned} \quad (53)$$

when the sign of the  $i\varepsilon$  terms is entirely determined by the prescriptions on the external legs. The Ward identities are easily obtained (we recall that one index is equal to  $A$  and all the others are equal to  $R$ )

$$Q_{1\mu} C_{\alpha\beta\gamma\delta}^{\mu\nu}(P, Q_1, Q_2, R) = V_{\delta \cdot (\alpha + \beta)}^\nu(R, Q_2, -R - Q_2) - V_{\alpha \cdot (\beta + \delta)}^\nu(P, Q_2, -P - Q_2) \quad (54)$$

where the symbol  $\cdot$  in the indices means that the corresponding R/A index need not be specified.

We turn now to the case of  $C_{AAAR}^{\mu\nu}(P, Q_1, Q_2, R)$  where we consider also the cyclic permutations on the indices. For this purpose it is useful to introduce the normalized functions  $\tilde{C}_{AAAR}^{\mu\nu}(P, Q_1, Q_2, R)$  defined by [18], [20]

$$C_{\alpha\beta\gamma\delta}^{\mu\nu}(P, Q_1, Q_2, R) = \frac{(n^F(p_0))^{\delta_{\alpha A}} (n^B(q_{10}))^{\delta_{\beta A}} (n^B(q_{20}))^{\delta_{\gamma A}} (n^F(r_0))^{\delta_{\delta A}}}{n(\delta_{\alpha A} p_0 + \delta_{\beta A} q_{10} + \delta_{\gamma A} q_{20} + \delta_{\delta A} r_0)} \tilde{C}_{\alpha\beta\gamma\delta}^{\mu\nu}(P, Q_1, Q_2, R) \quad (55)$$

which in the hard loop approximation is given by eq. (53). This can be seen either by direct calculation or by using the general relation [18] (only one index  $\alpha$ ,  $\beta$ ,  $\gamma$  or  $\delta$  is equal to R)

$$\tilde{C}_{\alpha\beta\gamma\delta}^{\mu\nu}(P, Q_1, Q_2, R) = (C_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}^{\mu\nu}(P, Q_1, Q_2, R))^* \quad (56)$$

The function  $\tilde{C}_{\alpha\beta\gamma\delta}^{\mu\nu}(P, Q_1, Q_2, R)$  therefore satisfies eq. (53) which holds true then whenever three of the indices are identical. We can go back to the full Green functions and we get, for example,

$$\begin{aligned} Q_{1\mu} C_{RRRA}^{\mu\nu}(P, Q_1, Q_2, R) &= \Gamma_{ARR}^\nu(R, Q_2, -R - Q_2) - \Gamma_{RAA}^\nu(P, Q_2, -P - Q_2) \\ Q_{1\mu} C_{AAAR}^{\mu\nu}(P, Q_1, Q_2, R) &= - \frac{n^F(p_0) n^B(q_{10})}{n^F(p_0 + q_{10})} \Gamma_{RAA}^\nu(R, Q_2, -R - Q_2) \\ &\quad + \frac{n^B(q_{10}) n^F(p_0 + q_{20})}{n^F(p_0 + q_{10} + q_{20})} \Gamma_{AAR}^\nu(P, Q_2, -P - Q_2) \end{aligned} \quad (57)$$

We note that in the second case there appears a pre-factor in front of the vertex function depending explicitly on the photon momentum  $Q_1$ : this is necessary since the 4-point function carries such factors whereas in the 3-point functions momentum  $Q_1$  does not appear. This suggests that, in general, Ward identities take a simpler form in terms of the normalized functions of type  $V_{\alpha\beta\gamma}^\mu$  and  $\tilde{C}_{\alpha\beta\gamma\delta}^{\mu\nu}$ , which are analytic continuations of the imaginary time formalism, than in terms of the full R/A functions. This can be verified when considering the more complicated case of  $C_{ARAR}^{\mu\nu}(P, Q_1, Q_2, R)$  which in the hard loop approximation reduces to

$$\begin{aligned} C_{ARAR}^{\mu\nu}(P, Q_1, Q_2, R) &= m_{th}^2(\epsilon) \int \frac{1}{2} \frac{d\hat{L}}{(2\pi)^{1-2\epsilon}} \hat{L}^\mu \hat{L}^\nu \hat{L} \frac{1}{P\hat{L} - i\epsilon} \frac{1}{R\hat{L} + i\epsilon} \\ &\quad \left( \frac{1 - n^F(p_0) - n^F(r_0 + q_{20})}{(P + Q_1)\hat{L} + i\epsilon} + \frac{1 + n^B(q_{20}) - n^F(-r_0 - q_{20})}{(P + Q_1)\hat{L} - i\epsilon} + \frac{1 - n^F(p_0) + n^B(q_{20})}{(P + Q_2)\hat{L} - i\epsilon} \right) \end{aligned} \quad (58)$$

It leads to the following Ward identity:

$$\begin{aligned} Q_{1\mu} C_{ARAR}^{\mu\nu}(P, Q_1, Q_2, R) &= \Gamma_{AAR}^\nu(P, Q_2, -P - Q_2) - \Gamma_{RAA}^\nu(R, Q_2, -R - Q_2) \\ &\quad + \frac{n^F(p_0) n^F(r_0 + q_{20})}{n^F(-q_{10})} \Gamma_{RAA}^\nu(R, Q_2, -R - Q_2) \end{aligned} \quad (59)$$

The last two terms appear because the prescription on the energy variable  $r_0 + q_{20}$  is not defined by the external conditions and in such a case the R/A formalism selects a particular linear combination of the retarded and advanced continuations.

## V. 2-point function in the multi-loop approximation

In the framework of the Braaten-Pisarski resummation we need the expression of Green's functions beyond the one-loop approximation. In particular, one is led to evaluate two-point functions with vertex corrections such as shown in Fig. 8a. We derive a general expression for the two-point function at the multi-loop level, independently of the hard loop approximation to which we return at the end of the section.

In the case of interactions involving only three particle vertices, it is always possible to cut the diagram in such a way as to have only two particle intermediate states (Fig. 8b). This allows us to express the self-energy in terms of the vertex functions  $\Gamma_{\alpha\beta\delta}$

$$-i\Gamma_{\beta\beta'}^{(j+k+1)}(Q) = -e^2 \int \frac{d^n P}{(2\pi)^n} \Delta_\alpha(P) \Gamma_{\alpha\beta\delta}^{(j)}(P, Q, -R) \Delta_\delta(R) \Gamma_{\bar{\alpha}\bar{\beta}'\delta}^{(k)}(-P, -Q, R) \quad (60)$$

where the various  $\Gamma$  functions carry the superscripts  $(j)$ ,  $(k)$ , and  $(j+k+1)$  to denote the number of loops at which they have been respectively evaluated. Introducing the normalized functions as in eq. (28) and regrouping terms we obtain (dropping irrelevant spin factors)

$$\begin{aligned} -i\Gamma_{\beta\beta}^{(j+k+1)}(Q) = & -e^2 \int \frac{d^n P}{(2\pi)^n} \\ & \left\{ \left( \frac{1}{2} + \eta_P n^{[\eta_P]}(p_0) \right) \text{Disc}_P \left[ \Delta_R(P) V_{R\beta A}^{(j)}(P, Q, -R) \Delta_R(R) V_{A\bar{\beta}R}^{(k)}(-P, -Q, R) \right] \right. \\ & \left. + \left( \frac{1}{2} + \eta_R n^{[\eta_R]}(r_0) \right) \text{Disc}_R \left[ \Delta_A(P) V_{A\beta A}^{(j)}(P, Q, -R) \Delta_R(R) V_{A\bar{\beta}R}^{(k)}(-P, -Q, R) \right] \right\} \end{aligned} \quad (61)$$

where the symbol  $\text{Disc}_P$  applied on a function  $F_{\alpha\beta\delta}(P, Q, R)$  means taking the discontinuity in the energy variable  $p_0$  of the function as defined by

$$\text{Disc}_P F_{R\beta\delta}(P, Q, R) = F_{R\beta\delta}(P, Q, R) - F_{A\beta\delta}(P, Q, R) \quad (62)$$

(see the discussion around eq. (50)). The ingredients to derive this relation are the same as those to obtain the one loop expression, namely the diagonality of propagators, the validity of eq. (27) for the dressed vertex and the property that terms with poles only on one side of the real axis of the loop energy variables give a vanishing contribution when no statistical weight, depending on the loop energy variable, is attached to them. Despite a rather cumbersome notation, the structure of eq. (60) is rather simple and similar to the one-loop result. Consider, as an example, the calculation of  $\Gamma_{RR}(Q)$ . The function in the first line, whose  $\text{Disc}_P$  should be evaluated, is trivially obtained with propagators and vertex

functions in the retarded  $P$  and  $Q$  momenta (recall also the crossing relation eq. (13)). Momentum  $R$  is necessarily retarded because of momentum conservation. Likewise the second line is constructed from the retarded momenta  $Q$  and  $R$ . However,  $P$  is now of the advanced type since, using  $p_0 = r_0 - q_0$  and recalling that taking the discontinuity in  $r_0$  puts  $r_0$  as the real axis, the prescription for  $p_0$  is that of  $-q_0$ , *i.e.*, advanced. Taking the discontinuity of these products of propagators and vertices generates two types of terms:  $\text{Disc}_P \Delta_R(P) = 2\pi \varepsilon(p_0) \delta(P^2 - M^2)$  puts the  $P$  line on shell while  $\text{Disc}_P V_{R\beta\delta}(P, Q, -R)$  amounts to taking the cut contribution of the vertex.

Likewise it is easy to prove

$$\Gamma_{\beta\bar{\beta}}(Q) = 0.$$

As an application we may consider the case when  $V_{\alpha\beta\delta}$  consists in a one loop diagram and study  $\text{Disc}_P V_{\alpha\beta\delta}(P, Q, R)$ . We have already seen that the vertex is expressed as a sum of the diagrams (eq. (29) and fig. 5). Taking the discontinuity with respect to  $p_0$  means, according to eq. (62), putting the internal momentum lines carrying the index  $\alpha$  on shell. An examination of eq. (29), leads then to taking the line  $L_2 = L_1 + P$  on shell when  $L_1$  is cut and  $L_1 = L_2 - P$  on shell when  $L_2$  is cut. This we symbolize by a cross on the corresponding line (see fig. 9). We use a different symbol to denote this cut because the statistical weight associated to taking  $\text{Disc}_P$  is  $(\frac{1}{2} + \eta_P n^{[n_P]}(p_0))$  rather than the thermal factor appropriate for the internal line. Therefore,  $\text{Disc}_P V_{\alpha\beta\delta}$  in eq. (61) picks up the 2-particle cut contribution in  $p_0$  while  $\text{Disc}_P \Delta_R(P)$  selects the pole in  $p_0$ . One can easily convince one-self, by such arguments that the self-energy in the multi-loop approximation retains a tree structure as it does at one loop.

In the next section we will be led to consider the imaginary part of the two point functions which is evaluated by constructing  $\Gamma_{RR}(Q) - \Gamma_{AA}(Q)$ . It can be written as

$$\begin{aligned} \Gamma_{RR}(Q) - \Gamma_{AA}(Q) = & -ie^2 \int \frac{d^n P}{(2\pi)^n} \left\{ \left( \frac{1}{2} + \eta_P n^{[n_P]}(p_0) \right) \right. \\ & \left[ \text{Disc}_P \Delta_R(P) V_{RRA}(P, Q, -R) \Delta_R(R) V_{AAR}(-P, -Q, R) \right. \\ & \left. - \text{Disc}_P \Delta_R(P) V_{RAR}(P, Q, -R) \Delta_A(R) V_{ARA}(-P, -Q, R) \right] \\ & + \left( \frac{1}{2} + \eta_R n^{[n_R]}(r_0) \right) \\ & \left[ \text{Disc}_R \Delta_A(P) V_{ARA}(P, Q, -R) \Delta_R(R) V_{RAR}(-P, -Q, R) \right. \\ & \left. - \text{Disc}_R \Delta_R(P) V_{RAA}((P, Q, -R) \Delta_R(R) V_{ARR}(-P, -Q, R) \right] \left. \right\}. \end{aligned} \quad (63)$$

The difference of the  $\text{Disc}_P$  or  $\text{Disc}_R$  expressions above have a simple interpretation. Let us remark that in the first square brackets the two terms differ by the indices associated to momenta  $Q$  and  $R$  while in the second ones the terms differ by the indices associated to momenta  $Q$  and  $P$ . Consider the  $\text{Disc}_P$  case. As we have seen, taking the discontinuity



with respect to the variable  $p_0$  amount to putting the energy  $p_0$  on the real axis. By “momentum conservation”,  $r_0 = -p_0 - q_0$ , the retarded/advanced prescription on the internal momentum  $R$  is now entirely determined by that on the external momentum  $Q$ :  $r_0$ , in that sense, becomes a function of  $q_0$ . This is expressed by the fact that the  $\text{Disc}_P$  terms differ by their indices in both  $Q$  and  $R$ . Keeping this in mind, the square brackets then isolate the discontinuity in  $q_0$  of the considered functions and we may write

$$\begin{aligned} & \text{Disc}_Q \text{Disc}_P \Delta_R(P) V_{RRA}(P, Q, -R) \Delta_R(R) V_{AAR}(-P, -Q, R) \\ &= \text{Disc}_P \Delta_R(P) V_{RRA}(P, Q, -R) \Delta_R(R) V_{AAR}(-P, -Q, R) \\ &- \text{Disc}_P \Delta_R(P) V_{RAR}(P, Q, -R) \Delta_A(R) V_{ARA}(-P, -Q, R) \end{aligned} \quad (64)$$

where  $R$  is considered as a function of  $Q$ , when  $\text{Disc}_P$  is evaluated.

If we turn now to the hard loop approximation we find that the singularity structure of the effective vertices simplifies considerably. A case in point is the QED vertex studied in the previous section where it was shown that (in the hard loop approximation)  $\text{Disc}_Q V_{\alpha R \delta}(P, Q, R) = 0$  (eq. (50a)). This function enters the calculation of the vacuum polarization tensor  $\pi_{\beta}^{\mu\nu}(Q)$  to be considered shortly. In eq. (64), as a consequence, we can write  $\text{Disc}_R \text{Disc}_P$  for the first term and  $\text{Disc}_P \text{Disc}_R$  for the second one. Furthermore, from eq. (49) it appears that the dependence on  $P$  and  $R$  factorizes in the integrand which we can write as a product of two functions  $f_{\alpha}^{(1)}(P) f_{\delta}^{(2)}(R)$ : the double discontinuity becomes then a product of discontinuities so that

$$\begin{aligned} \Gamma_{RR}(Q) - \Gamma_{AA}(Q) = -ie^2 \int \frac{d^n P}{(2\pi)^n} \left\{ (1 - n^F(p_0)) \text{Disc}_P f_R^{(1)}(P) \text{Disc}_R f_R^{(2)}(R) \right. \\ \left. + ((1 - n^F(r_0)) \text{Disc}_P f_A^{(1)}(P) \text{Disc}_R f_R^{(2)}(R)) \right\} \end{aligned} \quad (65)$$

where we have considered the case with an internal fermion loop. Combining both terms and using the detailed balance relation

$$n^F(r_0) - n^F(p_0) = (1 - e^{\beta q_0}) n^F(r_0) n^F(-p_0)$$

it comes out

$$\begin{aligned} \Gamma_{RR}(Q) - \Gamma_{AA}(Q) = -ie^2 \int \frac{d^{n-1} P}{(2\pi)^{n-1}} (1 - e^{\beta q_0}) \int dp_0 dr_0 n^F(r_0) n^F(-p_0) \delta(q_0 + p_0 - r_0) \\ \frac{1}{2\pi} \text{Disc}_P f_R^{(1)}(P) \text{Disc}_R f_R^{(2)}(R). \end{aligned} \quad (66)$$

This is to be compared to the formula in [5], [6] which expresses the discontinuity of the two-point function as an integral over real energies

$$\begin{aligned} & \int \frac{d^{n-1} P}{(2\pi)^{n-1}} \text{Disc } T \sum_{p_0} f^{(1)}(p_0, \vec{p}) f^{(2)}(q^0 - p^0, \vec{q} - \vec{p}) \\ &= i \int \frac{d^{n-1} P}{(2\pi)^{n-1}} (1 - e^{\beta q_0}) \int d\omega d\omega' n^F(\omega') n^F(\omega') \delta(q_0 - \omega - \omega') 2\pi \rho_1(\omega, \vec{p}) \rho_2(\omega', \vec{q} - \vec{p}) \end{aligned} \quad (67)$$

where the spectral density  $\rho_i$  is defined as, for example,  $\rho_i(\omega, \vec{p}) = 2\pi \text{Disc } f^{(i)}(p_0, \vec{p})$ .

The structure of eqs. (61), (63) is not changed, if instead of the bare propagators  $\Delta_\alpha(P)$ ,  $\Delta_\delta(R)$  we use in these equations the effective propagators  $^*\Delta_\alpha(P)$ ,  $^*\Delta_\delta(R)$ . The poles near the real axis are shifted away from it and become singularities in the same half-plane of the loop energy variable. In particular, the crucial ingredient in deriving these equations, namely that an advanced propagator has no pole in the lower half-plane and a retarded one has no pole in the upper half-plane, is not affected. When evaluating the discontinuities of these effective propagators we will not only get the pole contributions as in the bare case but also two particle cuts associated to the Landau damping mechanism.

## VI. Real soft photon production in a quark-gluon plasma.

The problem of photon emission in a quark-gluon plasma has already been considered for several cases: soft virtual photon at rest [5] or moving [6],[25] and hard real photon [7],[8]. We apply the above formalism to soft real photon [26]. We assume massless quarks and we introduce the strong interactions coupling, denoted  $g$ , assuming  $g \ll 1$ . The photon production rate [27], assuming  $q_0$  positive for simplicity,

$$q_0 \frac{d\sigma}{d^3q} = -\frac{1}{(2\pi)^3} n_B(q_0) \text{Im} \Pi_\mu^\mu(Q)|_{\text{Retarded}} \quad (68)$$

is related to the trace of the polarization tensor which in the following we denote for short  $\Pi_R(Q)$ , after summing over the photon polarization states.

Consider first the one-loop contribution to  $\text{Im } \Pi_R(Q)$  before resummation, i.e. using bare propagators and vertices. Using eq. (18), it is trivially found that  $\text{Im } \Pi_R(Q)$  vanishes (but  $\text{Re } \Pi_R(Q) \neq 0$ ). The reason for this is simple: the only kinematical configurations possibly contributing to  $\text{Im } \Pi_R$  are the collinear decay of the photon into a  $q\bar{q}$  pair or the collinear emission or absorption of the photon by a quark or an antiquark in the plasma. However helicity conservation at the  $\gamma q\bar{q}$  vertex forbids these processes.

We turn now to the effective theory and evaluate the same "one loop" diagram using effective propagators and vertices as shown in Fig. 10 a). The tadpole diagram (Fig. 10 b) vanishes because it is traceless [5]. In the framework of the resummed perturbative series, the self-energy like diagram gives the dominant contribution to the production of soft virtual photons. We have

$$\begin{aligned} -i\Pi_R(Q) = & -e^2 \int \frac{d^n P}{(2\pi)^n} (1 - 2 n^F(p_0)) g_{\mu\nu} \\ & \text{Disc}_P \text{Tr} \left[ ^*\mathcal{S}_R(P) ^*V_{\text{RRA}}^\mu(P, Q, -R) ^*\mathcal{S}_R(R) ^*V_{\text{RRA}}^\nu(P, Q, -R) \right] \end{aligned} \quad (69)$$

Some words of explanation are required concerning this equation. The effective propagator  $^*\mathcal{S}_\alpha$  and vertex  $^*V_{\alpha\beta\delta}^\mu$  are understood to contain QCD corrections and not QED corrections. At the order at which we do the calculation it simply amounts to substituting  $e^2 \rightarrow C_F g^2$  in eq. (42) to take into account the change in coupling as well as the color factor. We thus define the thermal mass to be now

$$m_{th}^2 = C_F \frac{g^2 T^2}{8}, \quad (70)$$

all other equations in the previous section being unchanged. We have applied the property of eq. (31) to the effective vertex eq. (52) so that the same vertex function appears twice in the integrand above. Finally, the expected term proportional to  $(\frac{1}{2} - n_F(r_0))$  can be reduced to the one above after the change of variable  $r_0 = -p_0$  and the use of eqs. (31) and (43), hence the factor  $1 - 2 n^F(p_0)$ . Writing out the explicit form of  $*V_{\alpha\beta\delta}^\mu$  we thus have to calculate the diagrams of Fig. 11. We ignore the first one which does not present any difficulty and turn to the second (or equivalently the third) one which will be shown to present a collinear divergence. We have:

$$-i\Pi_R(Q)|_b = -e^2 m_{th}^2(\epsilon) \int \frac{d^n P}{(2\pi)^n} \int \frac{1}{2} \frac{d\hat{L}}{(2\pi)^{1-2\epsilon}} (1 - 2 n^F(p_0)) \text{Disc}_P \frac{\text{Tr}(*\mathcal{S}_R(P) \hat{L} * \mathcal{S}_R(R) \hat{L})}{(P\hat{L} + i\epsilon)(R\hat{L} + i\epsilon)} \quad (71)$$

We choose to carry the  $\int dp_0$  integration by closing the contour in the upper half-plane. Writing  $\text{Disc}_P$  explicitly as the difference of two terms,

$$\text{Disc}_P Tr = \frac{\text{Tr}(*\mathcal{S}_R(P) \hat{L} * \mathcal{S}_R(R) \hat{L})}{(P\hat{L} + i\epsilon)(R\hat{L} + i\epsilon)} - \frac{\text{Tr}(*\mathcal{S}_A(P) \hat{L} * \mathcal{S}_R(R) \hat{L})}{(P\hat{L} - i\epsilon)(R\hat{L} + i\epsilon)} \quad (72)$$

we see that no contribution arises from the first term since all its singularities are located below the real axis. On the contrary, the second one contains a pole in the upper half plane  $(1/(P\hat{L} - i\epsilon))$ , coming from the hard loop in the effective vertex as well as a singularity from the resummed propagator  $*\mathcal{S}_A(P)$ . Let us concentrate on the pole contribution. It comes out

$$-i\Pi_R(Q)|_b = ie^2 m_{th}^2(\epsilon) \int \frac{d^n P}{(2\pi)^{n-1}} \frac{1}{2} \int \frac{d\hat{L}}{(2\pi)^{1-2\epsilon}} \delta(P\hat{L}) (1 - 2n^F(p_0)) \frac{\text{Tr}(*\mathcal{S}_A(P) \hat{L} * \mathcal{S}_R(R) \hat{L})}{(Q\hat{L} + i\epsilon)} \quad (73)$$

where the  $p_0$  integration is understood now to run along the real axis. The denominator  $Q\hat{L} + i\epsilon$  is nothing but  $R\hat{L} + i\epsilon$  where the  $\delta$ -function constraint has been used. Both  $Q$  and  $\hat{L}$  being light-like this factor leads to a collinear divergence when the angular integration  $\int d\hat{L}$  is performed. From now on, we are only interested in this diverging part neglecting all finite terms in the calculation. To evaluate the residue at the pole, it is enough to set  $\hat{L} = \hat{Q} = Q/q$  in the above, except of course, in the denominator, leading to the rather simple expression

$$-i\Pi_R(Q)|_{b,sing} = ie^2 m_{th}^2(\epsilon) \int \frac{d^n P}{(2\pi)^{n-1}} \delta(P\hat{Q}) (1 - 2n^F(p_0)) \text{Tr}(*\mathcal{S}_A(P) \hat{Q} * \mathcal{S}_R(R) \hat{Q}) \frac{1}{2} \int \frac{d\hat{L}}{(2\pi)^{1-2\epsilon}} \frac{1}{Q\hat{L} + i\epsilon} \quad (74)$$

The angular integration is understood in  $n - 2$  dimensions ( $n = 4 - 2\epsilon$ ) and its real part has a pole in  $\epsilon$ . Therefore we define

$$\frac{\gamma(\epsilon)}{\epsilon} = -\frac{1}{2} \int \frac{d\hat{L}}{(2\pi)^{1-2\epsilon}} \frac{q}{Q\hat{L} + i\epsilon} \quad (75)$$

neglecting a finite imaginary piece. The diverging contribution from the two diagrams containing one hard loop effective vertex is therefore

$$\begin{aligned} \Pi_R(Q)|_{b+c, sing} = 2 e^2 \frac{m_{th}^2(\epsilon)}{q} \frac{\gamma(\epsilon)}{\epsilon} \int \frac{d^n P}{(2\pi)^{n-1}} \delta(P\hat{Q})(1 - 2n^F(p_0)) \\ \text{Tr}(*S_A(P) \hat{Q} *S_R(R) \hat{Q}) \end{aligned} \quad (76)$$

To sum up, the diverging contribution arises from the momentum configuration where both  $P\hat{L}$  and  $R\hat{L}$  vanish that is when the fermion propagators coupling to the external photon both approach their mass-shell condition. It is interesting and somewhat paradoxical that such a divergence is a property of the resummed theory which tells us to use the effective propagators  $*\Delta_\alpha(P)$  and  $*\Delta_\delta(R)$  rather than the bare ones. Had we used the latter, coming back to eq. (71), we would have found that the trace reduces to

$$\text{Tr}P\hat{L}R\hat{L} = 8P\hat{L} R\hat{L} \quad (77)$$

cancelling both poles at the origin of the collinear divergence.

It is worth noting that the collinear divergence in eq. (75) is related to the vanishing mass of the photon. If  $Q^2$  were slightly off-shell the pole in eq. (75) would never be reached in the time like case or would be defined through a principal value prescription in the space like case.

Let us turn now to the last diagram with two hard loop effective vertices. It is explicitly

$$\begin{aligned} -i\Pi_R(Q)|_d = -e^2 m_{th}^4(\epsilon) \int \frac{d^n P}{(2\pi)^n} \frac{1}{2} \int \frac{d\hat{L}_1}{(2\pi)^{1-2\epsilon}} \frac{1}{2} \int \frac{d\hat{L}_2}{(2\pi)^{1-2\epsilon}} \\ \text{Disc}_P \left( \frac{\hat{L}_1 \hat{L}_2 \text{Tr}(*S_R(P) \hat{L}_1 *S_R(R) \hat{L}_2)}{(P\hat{L}_1 + i\epsilon)(R\hat{L}_1 + i\epsilon)(P\hat{L}_2 + i\epsilon)(R\hat{L}_2 + i\epsilon)} \right) \end{aligned} \quad (78)$$

As in the previous case, a collinear divergence will arise from the collinear configuration  $\hat{L}_1 = \hat{Q}$ , leading to a pole in  $\epsilon$ . It is to be noted that the configuration  $\hat{L}_2 = \hat{Q}$ , together with the collinearity condition on  $\hat{L}_1$  does not lead to a double pole because of the  $\hat{L}_1 \hat{L}_2$  factor. Setting thus  $\hat{L}_1 = \hat{Q}$  in the denominator we construct in fact a combination of type  $Q_\mu V_{\alpha\beta\delta}^\mu(P, Q, -R)$  which can be immediately reduced via the Ward identity eq. (47). More precisely we have in the integrand

$$m_{th}^2(\epsilon) \frac{1}{2} \int \frac{d\hat{L}_2}{(2\pi)^{1-2\epsilon}} \frac{\hat{Q}\hat{L}_2 \hat{L}_2}{(P\hat{L}_2 + i\epsilon_\alpha)(R\hat{L}_2 + i\epsilon_\delta)} = \frac{1}{q} (\Sigma_\alpha(P) - \Sigma_\delta(R)). \quad (79)$$

Recasting the  $\int dp_0$  integration as in eq. (73) and exhibiting the collinear divergence it comes out

$$-i\Pi_R|_{d,sing} = -2 i e^2 m_{th}^2(\epsilon) \frac{\gamma(\epsilon)}{\epsilon} \int \frac{d^n P}{(2\pi)^{n-1}} \delta(P\hat{Q}) (1 - 2 n^F(p_0)) \frac{1}{q} \text{Tr}(*S_A(P) \hat{Q} *S_R(R) (\Sigma_A(P) - \Sigma_R(R))) \quad (80)$$

The factor 2 is introduced to account for the case when  $\hat{L}_2$  is collinear to  $\hat{Q}$ . Using the definition eq. (44) of the effective propagator the trace term is reduced to

$$i (\text{Tr}(*S_A(P)\hat{Q}) - \text{Tr}(*S_R(R)\hat{Q})) - q \text{Tr}(*S_A(P) \hat{Q} *S_R(R) \hat{Q}) \quad (81)$$

It is immediately apparent that the last term exactly compensates the single effective vertex pole contribution eq. (76). We are thus left, adding all pieces together, with

$$\Pi_R(Q)|_{sing} = i 2 e^2 \frac{m_{th}^2(\epsilon)}{q^2} \frac{\gamma(\epsilon)}{\epsilon} \int \frac{d^n P}{(2\pi)^{n-1}} \delta(P\hat{Q}) (1 - 2n^F(p_0)) (\text{Tr}(*S_A(P) \hat{Q}) - \text{Tr}(*S_R(R) \hat{Q})) \quad (82)$$

The  $\Pi_A(Q)$  2-point function is easily deduced from this equation by replacing  $*S_\alpha(P)$  by  $*S_{\bar{\alpha}}(P)$  and the same for  $*S_\delta(R)$ . We introduce now the usual parametrization of the effective fermion propagator

$$*S_R(P) = i \sum_{s=\pm 1} \frac{\hat{P}_s}{D_R^s(p_0 + i\epsilon, \vec{p})} \quad (83)$$

where  $\hat{P}_s$  is the light-like vector  $\hat{P}_s = (1, s\hat{p})$  and  $D^s(P)$  is a scalar function whose expression can be found in [5], [6]. The sum runs over the two propagating modes of the thermalized fermion. From the general property of 2-point functions, eq. (21), and the definition eq. (44), we can write

$$*S_A(P) = -(*S_R(P))^{c.c} = i \sum_s \frac{\hat{P}_s}{(D_R^s(p_0 + i\epsilon, \vec{p}))^*} \quad (84)$$

which expresses the advanced effective propagator as the complex-conjugate of the retarded one. It is customary to use the notation

$$\frac{1}{D_{R,A}^s} = \alpha_s(P) \mp i\pi\beta_s(P) \quad (85)$$

We can now construct the imaginary part of the polarization function

$$\text{Im}\Pi_R(Q) = 2 e^2 \frac{m_{th}^2(\epsilon)}{q^2} \frac{\gamma(\epsilon)}{\epsilon} \int \frac{d^n P}{(2\pi)^{n-1}} \delta(P\hat{Q}) (1 - 2n^F(p_0)) \pi \sum_s (\beta_s(P) \text{Tr}\hat{P}_s\hat{Q} + \beta_s(R) \text{Tr}\hat{R}_s\hat{Q}) \quad (86)$$

The traces are easily evaluated and give  $4(1 - s\hat{p}\hat{q})$  and  $4(1 - s\hat{r}\hat{q})$  respectively, which when taking into account the  $\delta$ -function constraint in the integrand reduced to  $4(1 - sp_0/p)$  and  $4(1 - sr_0/r)$  leading to

$$\text{Im}\Pi_R(Q) = 8 e^2 \frac{m_{th}^2(\epsilon)}{q^2} \frac{\gamma(\epsilon)}{\epsilon} \int \frac{d^n P}{(2\pi)^{n-1}} \delta(P\hat{Q}) (1 - 2n^F(p_0)) \pi \sum_s \left( \left(1 - \frac{sp_0}{p}\right) \beta_s(P) + \left(1 - \frac{sr_0}{r}\right) \beta_s(R) \right) \quad (87)$$

If we remember that the  $\beta_s$  function are proportional to the same  $(1 - sp_0/p)$  type factor with a positive coefficient we see that the integral does not vanish and therefore the soft fermion loop contribution to the  $\gamma$  production rate in a plasma is divergent. It is interesting to note that the condition  $\delta(P\hat{Q})$  enforces that  $P$  (and also  $R$  since  $\delta(R\hat{Q}) = \delta(P\hat{Q})$ ) be space like ( $|p_0| < p$ ), ( $|r_0| < r$ ). The residue of the collinear pole is entirely due to the Landau damping contribution to the fermion effective propagator.

An alternative derivation of eq. (87) has recently been given in the imaginary time formalism [26]. It seems appropriate now to contrast the two approaches. Since we work here in the real time formalism we can use the usual Dirac algebra familiar from the  $T = 0$  case and we do not have the added complication to redefine the algebra in Euclidean space. A more important point derives from the fact that we calculate the full retarded (or advanced) 2-point function and not only its imaginary part. We manipulate 4-dimensional, or rather  $n$ -dimensional integrals and the integrands, except for statistical weights, keep a covariant form very similar to the  $T = 0$  case (see *e.g.*, eqs. (71) and (78)): because we work in  $n$ -dimensions we can evaluate the integral over the energy by closing contours in the appropriate half-planes, as we do in eq. (71) and eq. (78), to pick-up only the pole contributions which lead to the singular behavior; since after this manipulation the integrand keeps its covariant form we easily use the Ward identity (eq. (47)) to extract the final result. In contrast, in the ITF approach, the imaginary part of the 2-point function is directly calculated as an integral of spectral functions over the real energy axis: the various pieces of the integrand have to be decomposed into their principal values and  $\delta$ -functions to extract the divergent behavior and since the covariant form is lost in favor of the spectral functions the Ward identity cannot be directly used to show the partial compensation between the different terms. An explicit calculation of all terms is necessary to obtain the final results.

## VII. Conclusions

In this work, we have pursued the study of the R/A formalism and derived, in particular, an important equation expressing the discontinuity of the 2-point function at any loop order. We have also formulated, in this approach, the hard loop expansion for the case of QED. We used the R/A formalism to calculate the rate of real soft phton production in a QCD plasma and found the result (eq. (87)) to be divergent in agreement with a recent calculation performed in the imaginary time formalism [26]. Compared to that

work we find the calculation in the  $R/A$  formalism somewhat simpler, since, using contour integration, we can extract the relevant contributions of poles in the complex energy plane. We avoid distinguishing between principal part and  $\delta$ -function contributions of a pole as when the integration is performed on the real axis. Also the cancellation of the single hard vertex loop diagrams with the double hard vertex loop diagram is clearly attributed to the thermal Ward identity eq. (47). The structure of the remaining diverging term involves the spectral function of the effective soft fermion propagator. It is interesting to see how, in the framework of the effective theory diagrams with different number of loops partially compensate one another.

There are several ways the divergent result of eq. (87) could be regularized. In [26] it was proposed to introduce a soft cut-off of order  $gT$ . Another possibility is to include corrections to the hard internal fermion lines which are the cause of the collinear divergence when they approach the mass-shell condition. This is in the spirit of several attempts to define the damping rate of a hard fermion. For example, including a width of  $\mathcal{O}(g^2T)$  on the hard internal fermion lines [28] would displace the poles away from the real axis and replace the  $1/\epsilon$  divergence by  $\ln(1/g)$ . Finally, we have considered in our calculation only the contribution of diagrams with two internal soft fermion lines (see Fig.11). As discussed above the hard fermion loop is kinematically allowed but it does not contribute at the lowest order using bare propagators because of helicity conservation at the vertices. Introducing corrections to hard loops avoids this constraint. It would be interesting to compare the order of such contributions to the result of eq. (87).

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## Figure Captions

- Fig. 1 The general "real-time" contour in the complex time plane.
- Fig. 2 The QED vertex in the R/A formalism : a) with an incoming fermion and an incoming anti-fermion; b) with an incoming fermion and an outgoing fermion.
- Fig. 3 The generic two-point function.
- Fig. 4 The generic three-point function.
- Fig. 5 The three-point function as a sum of tree diagrams. The / on a line means that the line is put on mass-shell and carries a statistical factor:  $\epsilon(l_{i0})(\frac{1}{2} + \eta_i n^{[\eta_i]}(l_{i0}))\delta(L_i^2)$ .
- Fig. 6 The fermion self-energy diagram.
- Fig. 7 The photon-photon-quark-antiquark four-point function in QED.
- Fig. 8 A diagram in the effective theory: the two-point function with vertex corrections; a) one loop vertices; b) dressed vertices.
- Fig. 9 Taking the discontinuity with respect to  $p_0$  of a vertex up to one loop. The cross represents the action of taking the discontinuity (i.e. putting the corresponding line on shell).
- Fig. 10 The soft fermion loop contribution to the rate of soft real photon production in a QCD plasma.
- Fig. 11 The same as Fig. 10 displaying the structure of the effective vertices.